Question 4.1: The broken generators

Below Eq. (7.18) we make the statement that our choice of VEV keeps $T_3 + Y$ unbroken, while the other three generators are broken. Here you are asked to prove this statement.

1. With the choice of VEV in Eq. (7.18), show that $T_3 + Y$ annihilates the vacuum. That is, write $T_3 + Y$ explicitly as a 2×2 matrix, apply it to the vector $\langle \phi \rangle$, and show that the result is zero.

   This result proves that this operator is not broken.

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}. \tag{7.18}$$

2. The other three generators correspond to T_1, T_2, and $T_3 - Y$. Show that any non-vanishing linear combination of them (with real coefficients) does not annihilate the vacuum. This is a proof that there is no other unbroken generator.

Question 4.2: The would-be Nambu-Goldstone bosons

The spontaneous breaking of a global symmetry entails Nambu–Goldstone bosons. If we now modify the model such that the spontaneously broken symmetry is a local one, these scalar DoF become the longitudinal components of the gauge bosons that correspond to the broken generators. To see this explicitly, we consider the Higgs sector of a modified LSM, such that the spontaneously broken symmetry is global.

We consider the Higgs potential of Eq. (7.15).

$$-\mathcal{L}_\phi = \mu^2 \left(\phi^\dagger \phi\right) + \lambda \left(\phi^\dagger \phi\right)^2 . \tag{7.15}$$

Instead of Eq. (7.21)

$$\phi(x) = \frac{1}{\sqrt{2}} \exp\left[\frac{i\left(\sigma_a \theta_a(x) - I\theta_3(x)\right)}{2v}\right] \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}. \tag{7.21}$$

we write ϕ as

$$\phi = \begin{pmatrix} g^+ \\ (v + h + ig^0)/\sqrt{2} \end{pmatrix} \tag{7.116}$$

where $g^+ = (\phi_1 - i\phi_2)/\sqrt{2}$.

1. Write $\phi^\dagger$ in terms of g^-, g^0 and h. Recall that $g^- = (g^+)^\dagger$.

2. Plug ϕ and $\phi^\dagger$ of Eq. (7.116) into Eq. (7.15) and find the spectrum. One way to get the spectrum is to define the mass-squared matrix as

$$M_{ij}^2 = \left.\frac{\partial^2 V}{\partial \phi_i \partial \phi_j^\dagger}\right|_{\phi_i = \phi_j^\dagger = 0}. \tag{7.117}$$

   Explicitly, for example,

$$M^2(g^\pm) = \left.\frac{\partial^2 V}{\partial g^+ \partial g^-}\right|_{g^+ = g^- = 0}. \tag{7.118}$$

3. Check that the spectrum has three massless DoF, the $g^\pm$ and g^0, and one massive DoF, h. Check that the mass of h is the same in the global and local cases.

Question 4.3: Lepton decays

1. Find in the PDG the main decay mode of the muon, and obtain the corresponding decay width.

2. Draw the tree level Feynman diagram for the leading muon decay.

3. Find in the PDG the bound on

$$\text{BR}(\mu \to e\gamma). \tag{7.119}$$

   What is the LSM prediction for this mode? Explain.